On Convergence Generation in Computing the Electro-Magnetic Casimir Force

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Z. Naturforsch. **63a,** 571 – 574 (2008); received March 4, 2008

We tackle the very fundamental problem of zero-point energy divergence in the context of the Casimir effect. We calculate the Casimir force due to field fluctuations by using standard cavity radiation modes. The validity of convergence generation by means of an exponential energy cut-off factor is discussed in detail.

Key words: Casimir Effect; Zero-Point Energy; van der Waals Force.
PACS number: 03.70+k (theory of quantized fields)

1. Introduction

We refer to the Casimir effect [1] as the attractive force between a pair of parallel uncharged metallic plates in vacuum, due to vacuum field fluctuations. At present there is renewed interest in the subject as revealed by the numerous references quoted in the recent book by K. A. Milton [2]. The difficulty in treating this effect lies in the fact that it is intimately related to the zero-point energy $\varepsilon = \hbar \sum_{\vec{k}} \frac{1}{2} \omega_{\vec{k}}$ [3] which, in principle, is infinite if a summation over all field modes $\vec{k}$ is performed. This awkward feature of the quantized electric fields leads to infinities in the treatment of the Casimir effect which have to be dealt with in an appropriate manner. In fact one argues that these infinities have no incidence on physically measurable quantities.

The force between the plates, being a measurable quantity, must be finite, but this is only the case if one postulates an opposite force generated by the field of the infnite space outside the cavity. In practice one calculates the zero-point energy inside the cavity with the help of some cut-off procedure and then derives the result with respect to the distance a between the plates. Here we consider in particular an energy cut-off produced by an exponential convergence generating factor as proposed by Fierz [4]. However, the force being represented by the derivative of the energy with respect to the parameter a, a problem arises from the fact that the cut-off in turn depends on this parameter as it is implicitly contained in the energy.

In the first part of this treatise we show that the energy cut-off procedure can nevertheless be applied to the force, as it splits the derivative into two parts: one finite and one diverging yet a-independent. The fact of the a-independence of the latter legitimates the assumption that it can be compensated by an external opposite force.

In the second part of our treatise we apply these conclusions to the electro-magnetic force calculated directly by introducing standard radiation field modes inside the cavity. The result not only yields the right a-dependence of the force but it integrates summation over polarization states as well.

2. Energy Truncation

If the zero-point energy is formally written as

$$\varepsilon = \frac{1}{2}\hbar \sum_{\vec{k}} \omega_k = \frac{1}{2}\hbar c \sum_{\vec{k}} k, \tag{1}$$

it is generally admitted that the Casimir force between parallel perfectly conducting plates is represented by the negative derivative

$$-\frac{\partial \varepsilon}{\partial a}, \tag{2}$$

where a is the distance between the plates. However, both quantities (1) and (2) are infinite, and methods have to be found for extracting the finite part of these quantities from their divergent part, assuming that the latter is compensated by the action of the infinite field outside the cavity.

Here we consider more specifically the use of a convergence generating factor $\mathrm{e}^{-\lambda k}$ as introduced by Fierz [4], where λ is an infinitesimal parameter. Using this factor, the quantity (1) is replaced by

$$\tilde{\varepsilon} = \frac{1}{2}\hbar c \sum_{\vec{k}} k \mathrm{e}^{-\lambda k}. \tag{3}$$

This method leads to an expression of the type $\tilde{\varepsilon} = \hbar c C/a^3$, where C is a numerical factor. Taking the derivative of this expression, one obtains for the force a law in a^{-4} which is indeed the correct relation. *However, formal derivation of a truncated quantity could seem questionable as shall be discussed below.*

In order to write out this expression more explicitly, we consider a configuration with the two conducting surfaces located at $z = 0$ and $z = a$ and extending to infinity in the x,y directions. Then the wave vector $\vec{k}$ has two continuous components, k_x,k_y, and a discontinuous one, $k_z = \frac{n\pi}{a}$, where n is an integer reaching from 1 to infinity. Thus we have $k^2 = \kappa^2 + \frac{n^2\pi^2}{a^2}$ with $\kappa^2 = k_x^2 + k_y^2$. If furthermore we make the replacement $\sum_{\vec{k}} \to \frac{1}{4\pi^2}\frac{1}{a}\sum_n \int \mathrm{d}^2\kappa$, (3) takes the form

$$\tilde{\varepsilon} = \frac{1}{2}\hbar c \frac{1}{4\pi^2}\sum_n \int \mathrm{d}^2\kappa k \mathrm{e}^{-\lambda k} \tag{4}$$

with

$$k = \sqrt{\kappa^2 + \frac{n^2\pi^2}{a^2}}. \tag{5}$$

The question now arises, whether the force, which is the only measurable quantity, can still be obtained from the derivative of $\tilde{\varepsilon}$, with respect to the distance a. This question is important and nontrivial, given the fact that the convergence generating factor $\mathrm{e}^{-\lambda k}$ depends on the parameter a, according to (5). In other words, is it legitimate to postulate the equivalence

$$\frac{\partial}{\partial a}\left(\sum_{\vec{k}} \frac{1}{2}\hbar c k \mathrm{e}^{-\lambda k}\right) \leftrightarrow \sum_{\vec{k}} \left\{\frac{\partial}{\partial a}\left(\frac{1}{2}\hbar c k\right)\mathrm{e}^{-\lambda k}\right\}\ ? \tag{6}$$

It is this point which has to be scrutinized carefully. For this purpose we consider the derivative of the expression (4) written in the form

$$\begin{aligned} -\frac{\partial\tilde{\varepsilon}}{\partial a} &= -\frac{1}{8\pi^2}\hbar c \sum_n \int \mathrm{d}^2\kappa \frac{\partial k}{\partial a}\mathrm{e}^{-\lambda k} \\ &\quad - \frac{1}{8\pi^2}\hbar c \sum_n \int \mathrm{d}^2\kappa (-\lambda)\frac{1}{2}\left(\frac{\partial}{\partial a}k^2\right)\mathrm{e}^{-\lambda k}. \end{aligned} \tag{7}$$

With k given by (5) we obtain for the first term of (7) the expression

$$T_1 = -\frac{1}{8\pi^2}\hbar c \sum_n \int \mathrm{d}^2\kappa \frac{-\frac{n^2\pi^2}{a^3}}{\sqrt{\kappa^2 + \frac{n^2\pi^2}{a^2}}}\mathrm{e}^{-\lambda\sqrt{\kappa^2 + \frac{n^2\pi^2}{a^2}}}. \tag{8}$$

We evaluate this quantity by making the following substitutions [4]:

$$\mathrm{d}^2\kappa = 2\pi\kappa \mathrm{d}\kappa, \quad \kappa^2 = \frac{n^2\pi^2}{a^2}z.$$

We thus obtain

$$T_1 = -\frac{\hbar c}{8\pi}\frac{1}{a}\sum_n \left(\frac{n^3\pi^3}{a^3}\right)\int_0^\infty \frac{-1}{\sqrt{z+1}}\mathrm{e}^{-\lambda\frac{n\pi}{a}\sqrt{z+1}}. \tag{9}$$

The integral can be calculated exactly with the result

$$T_1 = -\frac{\hbar c}{4\pi}\frac{1}{a}\sum_n \frac{n^2\pi^2}{a^2}\frac{-1}{\lambda}\mathrm{e}^{-\lambda\frac{n\pi}{a}}. \tag{10}$$

After generating the factor n^2 by deriving the exponential twice with respect to λ, the sum over n reduces to a geometric series. This yields the expression

$$T_1 = \frac{\hbar c}{4\pi}\frac{1}{\lambda}\frac{\partial^2}{\partial\lambda^2}\frac{1}{1-\mathrm{e}^{-\lambda\frac{\pi}{a}}}. \tag{11}$$

Introducing Bernoulli's number B_h defined by

$$\frac{-\lambda\frac{\pi}{a}}{\mathrm{e}^{-\lambda\frac{\pi}{a}}-1} = \frac{\lambda\frac{\pi}{a}}{1-\mathrm{e}^{-\lambda\frac{\pi}{a}}} = \sum_0^\infty \frac{B_h}{h!}(-1)^h\left(\frac{\pi}{a}\right)^h\lambda^h, \tag{12}$$

we expand (11) as follows:

$$T_1 = \frac{\hbar c}{4\pi}\frac{1}{a}\sum_0^\infty \frac{B_h}{h!}(-1)^h\left(\frac{\pi}{a}\right)^{h-1}(h-1)(h-2)\lambda^{h-4}. \tag{13}$$

Letting λ be an infinitesimal quantity, (13) reduces to

$$T_1 = \frac{\hbar c}{2\pi^2}\lambda^{-4} + \frac{\hbar c}{a^4}\pi^2\frac{6}{4\times 4!}B_4 \tag{14}$$

with $B_0 = 1$, $B_3 = 0$, $B_4 - \frac{1}{30}$.

We now turn to the second term in (7) denoted T_2. A similar, however somewhat lengthy calculation, leads to the result

$$\begin{aligned} T_2 = \frac{\hbar c}{4\pi}\frac{1}{a}\Bigg\{ &\sum_0^\infty \frac{B_h}{h!}(-1)^h\left(\frac{\pi}{a}\right)^{h-1} \\ &\cdot(h-1)(h-2)(h-4)\lambda^{h-4}\Bigg\}. \end{aligned} \tag{15}$$

In the limit of infinitesimal λ only the a-independent term in λ^{-4} survives, while the a-dependent term with $h = 4$ cancels. This proves the statement that, if unphysical a-independent diverging terms are disregarded, only the first term in (7) is significant, meaning that the equivalence of (6) is warranted.

This conclusion allows a straightforward calculation of the electro-magnetic Casimir force as we shall discuss now.

3. The Electro-Magnetic Casimir Force

In the appendix we derive an expression of the force F_z exerted on the $z = a$ boundary surface from inside the cavity: (A13). This expression is still diverging and one must therefore assume that the unphysical infinite part of it is compensated by an opposite force generated in the infinite space outside the cavity. In order to eliminate the divergencies, we first split the expression of F_z into two parts according to the relation

$$F_z = -\frac{\hbar c}{4\pi^2 a}\sum_n \int \mathrm{d}^2\kappa k - \frac{1}{4\pi^2}\frac{\hbar c}{a}\sum_n \int \mathrm{d}^2\kappa \frac{\frac{n^2\pi^2}{a^2}}{\sqrt{\kappa^2+\frac{n^2\pi^2}{a^2}}}. \tag{16}$$

Assuming that the relevant part of F_z must be proportional to the finite part of the scalar theory, it is clear that only the second term of (16) has to be considered. By comparing this term, with the convergence factor added, with (8), we see that it is equal to $2 \times T_1$. Consequently the value of the force is found to be two times the converging part of T_1 as given by (14). We thus arrive at the final result

$$F_z = 2\pi^2 \frac{\hbar c}{a^4}\frac{6}{4\times 4!}B_4 = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}, \tag{17}$$

which is the generally admitted value for the electro-magnetic Casimir force. In particular, it appears that by introducing the factor 2, the standard field modes integrate correctly the two polarization states of the radiation. However, the method does not demonstrate explicitly the way in which the action of the external field neutralizes the diverging terms in the expression of the force. This point has been treated by previous authors by means of more intricate calculations, e. g. the derivation presented in [2].

Note finally that the law in a^{-4} illustrates the more general fact that in order to obtain measurable quantities with no photons present, square field averages over small volumes have to be considered. By these procedures one obtains results $\propto \hbar c(\Delta l)^{-4}$ with Δl some appropriate linear dimension [5].

4. Conclusion

Convergence generation by means of an exponential factor, initially conceived for being applied to the zero-point energy, can be extended to the expressions of the electro-magnetic force if certain equivalences are taken into account. Then the force is obtained from standard cavity radiation modes [6] which automatically include summation over polarization states.

Acknowledgements

The author is indebted to Dr.'s Boris Oksengorn and Annie Spielfiedel for valuable scientific assistance.

Appendix

We consider again the cavity made of two plane parallel perfectly reflecting metallic plates located at distances $z = 0$ and $z = a$, respectively. We write the electric field mode functions inside the cavity in the form

$$\vec{E}(x,y,z,t) = \vec{E}\mathrm{e}^{\mathrm{i}\omega t}. \tag{A1}$$

In order to specialize these cavity modes, we choose bordering planes located at x or $y = 0, L$ and, as already mentioned, $z = 0, a$.

(i) At the boundaries the horizontal components of the electric field $\vec{E}$ should be zero in accordance with Maxwell's theory.

(ii) Inside the cavity the zero charge condition

$$\nabla \cdot \vec{E} = 0 \tag{A2}$$

should be fulfilled. A candidate satisfying both conditions has components given by the following expressions [6]:

$$\begin{aligned} E_x &= A_x \cos(k_x x)\sin(k_y y)\sin(k_z z), \\ E_y &= A_y \sin(k_x x)\cos(k_y y)\sin(k_z z), \\ E_z &= A_z \sin(k_x x)\sin(k_y y)\cos(k_z z), \\ \vec{k} &= \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} \frac{\pi n_x}{L} \\ \frac{\pi n_y}{L} \\ \frac{\pi n_z}{a} \end{pmatrix} \end{aligned} \tag{A3}$$

with n_x, n_y, n_z being integers $1, 2, \ldots$.

However, the zero charge condition (A2) imposes for the amplitudes the relation

$$k_x A_x + k_y A_y + k_z A_z = 0. \quad \text{(A4)}$$

A second relation for the amplitudes is obtained by linking the electro-magnetic energy density (i. e. twice the electric part) inside the cavity to the quantum mechanical zero-point energy and thus setting

$$2\varepsilon_0 \vec{E}^2 = \frac{1}{2}\frac{\hbar\omega_k}{L^2 a} \quad \text{(A5)}$$

with $\omega_k = ck$ and $k^2 = k_x^2 + k_y^2 + k_z^2$.

From (A3) one obtains by space integration for the average of the electric energy density the relation $\vec{E}^2 = \frac{1}{8}A^2$, so that (A5) reduces to

$$\varepsilon_0 2 \times \frac{1}{8}A^2 = \frac{1}{2}\frac{\hbar\omega_k}{L^2 a}$$

or

$$A^2 = A_x^2 + A_y^2 + A_z^2 = \frac{2}{\varepsilon_0}\frac{\hbar\omega_k}{L^2 a} = \frac{2}{\varepsilon_0}\frac{\hbar ck}{L^2 a}. \quad \text{(A6)}$$

Taking advantage of the x,y symmetry of (A4) and (A6), we set $A_x = k_x q$, $A_y = k_y q$ and write

$$\kappa^2 q + k_z A_z = 0, \kappa^2 q^2 + A_z^2 = A^2 \quad \text{(A7)}$$

with $\kappa^2 = k_x^2 + k_y^2$.

Solving these equations for A_z yields

$$A_z^2 = A^2 \frac{\kappa^2}{k^2} = \frac{2}{\varepsilon_0}\frac{\hbar ck}{L^2 a}\frac{\kappa^2}{k^2}, \quad \text{(A8)}$$

where (A6) has been used.

What we need is the force exerted on the plate at $z = a$ by the field present in the cavity. This problem of an electro-magnetic force in the absence of currents and space charges has a well-known solution. Note that it also arises in the classical theory of radiation pressure on a perfectly reflecting surface. In the present case the solution is given by

$$F_z = -\frac{1}{2}\varepsilon_0 \left(E_z(a)\right)^2. \quad \text{(A9)}$$

We now take the average over x, y. From the field expressions (A3) we thus find

$$\langle F_z \rangle = -\frac{1}{2}\cdot\frac{1}{4}\varepsilon_0 A_z^2, \quad \text{(A10)}$$

and, by substituting for A_z^2 the expression (A8),

$$\langle F_z \rangle = -\frac{1}{4}\frac{\hbar ck}{L^2 a}\frac{\kappa^2}{k^2}. \quad \text{(A11)}$$

This expression has been established for a given mode $\vec{k}$. In order to obtain the final result, we now have to sum over modes. The total force is therefore represented by

$$F_z = -\frac{1}{4}\frac{\hbar}{L^2 a}\sum_{\vec{k}} ck\frac{\kappa^2}{k^2}. \quad \text{(A12)}$$

We thus have to sum over mode numbers n_x, n_y, n_z. For $L \to \infty$ the $n_{x,y}$ summation can be replaced in the usual way by integration over elements $\frac{L}{\pi}\mathrm{d}k_{x,y}$ yielding in the end for the force the expression

$$F_z = -\frac{1}{4\pi^2}\frac{\hbar c}{a}\sum_n \int \mathrm{d}^2\kappa k\frac{\kappa^2}{k^2} \quad \text{(A13)}$$

with $n = n_z$.

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